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1997 J. Phys. A: Math. Gen. 30 7525

(http://iopscience.iop.org/0305-4470/30/21/023)

J. Phys. A: Math. Gen. **30** (1997) 7525–7535. Printed in the UK PII: S0305-4470(97)77506-1

Non-Abelian geometric phase for general three-dimensional quantum systems

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Received 28 August 1996

Abstract. Adiabatic $U(2)$ geometric phases are studied for arbitrary quantum systems with a three-dimensional Hilbert space. Necessary and sufficient conditions for the occurrence of the non-Abelian geometrical phases are obtained without actually solving the full eigenvalue problem for the instantaneous Hamiltonian. The parameter space of such systems which has the structure of $\mathbb{C}P^2$ is explicitly constructed. The results of this article are applicable for arbitrary multipole interaction Hamiltonians $H = Q^{i_1,\ldots,i_n} J_{i_1} \ldots J_{i_n}$ and their linear combinations for spin $j = 1$ systems. In particular it is shown that the nuclear quadrupole Hamiltonian $H = Q^{ij} J_i J_j$ does actually lead to non-Abelian geometric phases for $j = 1$. This system, being bosonic, is time-reversal invariant. Therefore, it cannot support Abelian adiabatic geometrical phases.

1. Introduction

In 1984 Berry published a beautiful article [1], in which he systematically studied what is now called the Berry phase or the adiabatic geometrical phase‡. Berry's observation has since attracted the attention of a large number of theoretical and experimental physicists. One of the most important developments in the subject has been the discovery of the non-Abelian analogues of the adiabatic geometrical phase by Wilczek and Zee [3]. This development has unravelled some interesting manifestations of the non-Abelian gauge theories in non-relativistic quantum mechanics, particularly in the realm of molecular physics [4].

Perhaps one of the most important results which contributed to a better understanding of the Abelian and non-Abelian geometrical phases is Simon's identification of Berry's Abelian phase with a holonomy element of a $U(1)$ spectral bundle over the space of the environmental parameters of the system, [5]. In the language of fibre bundles the non-Abelian phases of Wilczek and Zee correspond to holonomy elements of the $U(\mathcal{N})$ spectral bundle associated with an $\mathcal{N}$-fold degenerate eigenvalue of the Hamiltonian.

The range of the environmental parameters R, i.e. the structure of the parameter space M whose points are coordinated by $R = (R^1, \ldots, R^n)$, is determined by the condition of the stability of the degeneracy structure of the Hamiltonian, i.e. by the condition that, during any possible evolution of the parameters for any smooth curve $C : [0, T] \to M$, the degeneracy structure of the instantaneous Hamiltonian $H(t) = H[C(t)]$, must not change. Here one also assumes that the eigenvalues $E_n[R]$ and eigenvectors $|n; R\rangle$ of the Hamiltonian $H[R]$

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‡ Manifestations of the phenomenon of the geometric phase have been known to chemists [2] long before Berry's article.

are smooth functions of the parameters $R \in M$. Therefore, different degeneracy structures of the Hamiltonian correspond to distinct parameter spaces.

Although, there are by now hundreds of publications on geometric phases, the number of the specific examples which have been worked out in detail is quite few. The best-known examples which lead to Abelian geometric phases are as follows.

(1) Berry's original example of a magnetic dipole (a spin) in a rotating magnetic field with the Hamiltonian:

$$H[R] = b \sum_{i=1}^{3} R^i J_i \tag{1}$$

where b is a constant (Larmor frequency), R^i are the Cartesian coordinates of the tip of the magnetic field and J_i are angular momentum operators, i.e. generators of the dynamical group $SU(2)$, [1, 4]. The parameter space of this system is the two-dimensional sphere $S^2 = SU(2)/U(1)$.

(2) The generalized harmonic oscillator [6, 7], whose Hamiltonian can also be put in the form (1) by identifying J_i with the generators of $SU(1, 1)$, [8]. The parameter space of this system is the hyperbolic sphere $SU(1, 1)/U(1)$.

Immediate generalizations of these examples are obtained by taking an arbitrary dynamical group G and requiring the Hamiltonian H to belong to a unitary irreducible representation of the Lie algebra† $\mathcal{G}$ of G, i.e. choosing H to be a linear combination of (the representation of) the generators of G. For a compact semisimple group G, it is shown in [9], that the relevant parameter space is in general a subspace of the flag manifold G/T, where T is a maximal torus (largest Abelian subgroup) of G. This result holds for arbitrary quantum systems whose Hilbert space is finite-dimensional. For in this case the Hamiltonian is in general a finite-dimensional Hermitian matrix. Therefore it belongs to the Lie algebra $u(N)$ of the group $U(N)$, where N is the dimension of the Hilbert space.

For the non-Abelian phase, the known examples are as follows.

(3) The original example of Wilczek and Zee [3], which involves an $(\mathcal{N}+1)$-dimensional system with an $\mathcal{N}$-fold degeneracy. The Hamiltonian of this system is obtained by similarity transformations (rotations) by elements of $SO(\mathcal{N}+1)$ of a diagonal $(\mathcal{N}+1) \times (\mathcal{N}+1)$ matrix which has only one non-zero entry. Clearly, the system has an $SO(\mathcal{N})$ symmetry by construction. Hence the parameter space is $SO(\mathcal{N}+1)/SO(\mathcal{N}) = S^{\mathcal{N}}$.

(4) The fermionic (half odd-integer spin) systems associated with nuclear quadrupole Hamiltonians of the form:

$$H = Q^{ij} J_i J_j. \tag{2}$$

These systems which were first studied in the context of the geometric phase by Mead [10] and subsequently by Avron *et al* [11, 12], involve Kramers degeneracy. The parameter space is S^4. As shown in [12], for the bosonic case (integer spin), the Berry connection one-form is exact [13] (the Berry gauge potential is pure gauge). Therefore, the (Abelian) Berry phase angle vanishes. This is a direct consequence of the fact that the bosonic systems of this type are time-reversal invariant. The same conclusion cannot be reached for the non-Abelian case, however. The simplest nontrivial case is for spin $j = 1$, where the Hilbert space is three dimensional. This problem turns out to be also relevant to the manifestations of the geometric phase for relativistic scalar fields in Bianchi type IX cosmological backgrounds [14].

† One may equivalently say that H belongs to the Lie algebra of a unitary irreducible representation of the Lie group G. In this paper the representation of the Lie algebra of a Lie group will mean the natural representation induced by the representation of the group.

The study of the geometrical phase for three- and higher-dimensional Hilbert spaces is clearly plagued with the difficulties involving the solution of the eigenvalue problem for the instantaneous Hamiltonian. Even in the three-dimensional case, the characteristic polynomial is of order three which makes the direct solution of the eigenvalue problem quite complicated. The motivation for the present article has been the simple observation that the existence of a (two-fold) degeneracy can be exploited to identify the appropriate parameter space over which one can obtain the eigenvalues and the eigenvectors of the Hamiltonian straightforwardly. Unlike the direct approach, in which one tries to solve the full eigenvalue problem for the Hamiltonian and then obtain the parameter space by equating two of the eigenvalues, this method only involves the solution of simple quadratic equations. This is demonstrated in section 2. This section also includes a detailed discussion of the parameter space. Section 3 includes the computation of the geometric phases. The general results are then applied to the quadrupole and multipole Hamiltonians (2) in section 4.

In the remainder of this section, a brief review of the non-Abelian geometrical phase of Wilzcek and Zee is given.

Berry's investigation of the adiabatic geometrical phase uses the quantum adiabatic theorem [15]. If the time-dependence of the Hamiltonian $H(t) = H[R(t)]$ justifies the validity of the adiabatic approximation, an initial-state vector which is an eigenvector of the initial Hamiltonian $H(0) = H[R(0)]$, evolves according to the Schrödinger equation, in such a way that it always remains an eigenvector of the instantaneous Hamiltonian $H(t) = H[R(t)]$. If the evolving-state vector corresponds to an $\mathcal{N}$-fold degenerate eigenvalue $E_n[R(t)]$, then the adiabatic theorem states that it must always belong to the $\mathcal{N}$-dimensional degeneracy subspace $\mathcal{H}_n[R(t)]$ associated with $E_n[R(t)]$. If the Hamiltonian depends periodically on time, i.e. the curve $C : [0, T] \to M$ is closed, then after a period, the Hamiltonian, its eigenvalues, and the corresponding degeneracy subspaces return to their original form, i.e. $H(T) = H[R(T)] = H[R(0)] = H(0)$, $E_n(T) = E_n[R(T)] = E_n[R(0)] = E_n(0)$, and $\mathcal{H}_n(T) = \mathcal{H}_n[R(T)] = \mathcal{H}_n[R(0)] = \mathcal{H}_n(0)$. Therefore the evolving-state vector $|\psi(T)\rangle$ belongs to the same degeneracy subspace as the initial-state vector $|\psi(0)\rangle$. Since the evolution is supposed to be unitary, there exists a $U(\mathcal{N})$ matrix relating $|\psi(T)\rangle$ and $|\psi(0)\rangle$, which is given by [3]:

$$e^{-\frac{i}{\hbar}\int_0^T E_n(t)}\mathcal{P}[e^{i\oint_C A_n}]. \tag{3}$$

Here $\mathcal{P}$ is the path-ordering operator [13] and A_n is a $u(\mathcal{N})$-valued (connection) one-form whose matrix elements are locally given by:

$$A_n^{ab} = i\left\langle n, a; R \left| \frac{\partial}{\partial R^i} \right| n, b; R \right\rangle dR^i = i\langle n, a; R|d|n, b; R\rangle. \tag{4}$$

In this equation a and b are degeneracy labels, $\{|n, a; R\rangle\}$, with $a = 1, \ldots, \mathcal{N}$, is an orthonormal local basis of $\mathcal{H}_n[R]$, and d denotes the exterior derivative operator on M.

The first exponential in (3) is called the *dynamical phase*, whereas the second (path-ordered) exponential is called the *non-Abelian adiabatic geometrical phase*. As seen from (3), unlike the dynamical phase, the geometrical phase only depends on the shape of the curve C in M and not on its parametrization. Usually this property of the geometric phase is offered as a justification of its geometrical nature. This point of view does not, however, do justice to the most intriguing geometrical properties of this phase which are best described in terms of the geometry of spectral bundles on the space of parameters and the universal classifying bundles on the projective Hilbert space. For a thorough discussion of the mathematical structure of the geometric phase see [16, 9] and references therein.

Finally let me emphasize that the geometrical phase is not a topological quantity in general. By definition a topological phase, such as the Aharonov–Bohm phase, is invariant under smooth deformations of the curve C. This is not generally the case for arbitrary geometrical phases. Topological phases of this type form a proper subset of all geometrical phases. There have been some arguments in the literature concerning the topological content of the geometrical phase [17] (see also [8, 18]) in which the removal of a geometrical phase via a smooth deformation of the functional dependence of the Hamiltonian on the parameters, or through a time-dependent canonical transformation, has been used to justify the attribution of the term 'trivial' to these phases. A typical example of this type of removable geometrical phase occurs for the generalized harmonic oscillator. For a specific physical system, however, such deformations or canonical transformations cannot be freely affected. In this sense topologically trivial geometrical phases such as those associated with the generalized harmonic oscillator are as physically significant as their topologically nontrivial counterparts. Therefore, in this article, I shall use the phrase 'trivial geometrical phase' to mean that the corresponding matrix-valued 'phase angle' vanishes, i.e. the geometrical phase does not exist. A precise characterization of the topological content of the Abelian adiabatic geometrical phases which are associated with compact semisimple dynamical groups (finite-dimensional Hilbert spaces) is given in [9].

2. Three-dimensional systems

The simplest non-Abelian geometric phases belong to $U(2)$. Hence the first non-trivial case which leads to non-Abelian $U(2)$-geometric phases is when the dimension of the Hilbert space is three. In this case, the Hamiltonian is a 3×3 Hermitian matrix which can be viewed as an element of the Lie algebra $u(3)$, i.e.

$$H[R] = b \sum_{i=0}^{8} R^i \lambda_i \tag{5}$$

where R^i are real parameters and λ_i are generators of $U(3)$ in the defining representation. For example one can take λ_0 to be the identity matrix $I_{3\times 3}$, and identify $\lambda_1, \ldots, \lambda_8$ with the Gell–Mann matrices [19]. In the remainder of this article a unit system is used where $b = 1$. The adiabaticity assumption may then be expressed as $T \gg 1$.

It is shown in [9], that as a consequence of some group theoretical considerations the parameter space M is in general (a subspace of) the manifold $SU(3)/U(2) = \mathbb{C}P^2$. In this section, the parameter space for the case where one of the eigenvalues of the Hamiltonian is doubly degenerate will be explored. For this particular case it is quite straightforward to see that indeed $M = \mathbb{C}P^2$. Let us first express the Hamiltonian H in the form:

$$H[R] = \mathcal{U}[R] H_D \mathcal{U}^{-1}[R] \tag{6}$$

where $\mathcal{U}[R] \in U(3)$ and H_D is diagonal:

$$H_D = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_2 \end{pmatrix}. \tag{7}$$

In $U(3)$ there are two distinct $U(1)$ and $U(2)$ subgroups which respectively leave the degeneracy subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ invariant. Hence the true parameter space is $U(3)/[U(2) \times U(1)] = \mathbb{C}P^2$. This argument does not, however, provide a concrete characterization of the parameter space unless one actually solves the full eigenvalue equation for the Hamiltonian, i.e. finds the explicit expression for $\mathcal{U}[R]$ in terms of R^i. The main result of this paper is the fact that, due to the presence of the degeneracy, this procedure can be replaced with another, much more manageable, method which leads to an explicit construction of the parameter space and a direct computation of the associated geometric phases.

In order to implement the condition on the degeneracy, let me write the Hamiltonian in the form:

$$H = \begin{pmatrix} r & \xi^* & \zeta^* \\ \xi & s & \kappa^* \\ \zeta & \kappa & t \end{pmatrix} \tag{8}$$

where r, s, t are real and ξ, ζ, κ are complex parameters related to R^i, according to:

$$\begin{aligned} &r = R^0 + R^3 + R^8/\sqrt{3} \qquad s = R^0 - R^3 + R^8/\sqrt{3} \qquad t = R^0 - 2R^8/\sqrt{3} \\ &\xi = R^1 + \mathrm{i}R^2 \qquad \zeta = R^4 + \mathrm{i}R^5 \qquad \kappa = R^6 + iR^7. \end{aligned} \tag{9}$$

These equations are obtained using the expression for the Gell–Mann matrices, as given for example in [19], in equation (5) and comparing with equation (8).

Furthermore, since the addition of a multiple of the identity operator does not have any physical implications one can alternatively consider the Hamiltonian

$$H' := H - E_2 I_{3\times3} = \begin{pmatrix} r' & \xi^* & \zeta^* \\ \xi & s' & \kappa^* \\ \zeta & \kappa & t' \end{pmatrix} \tag{10}$$

where

$$r' = r - E_2 \qquad s' = s - E_2 \qquad t' = t - E_2. \tag{11}$$

Clearly H and H' have identical eigenvectors. Their eigenvalues are related by $E_1' = E_1 - E_2$ and $E_2' = 0$, where E_2' corresponds to the degenerate eigenvalue. Clearly, it is E_1' and the common eigenvectors which are physically significant†.

Next consider the eigenvalue problem for the Hamiltonian H'. The corresponding characteristic polynomial is given by:

$$\begin{aligned} P(E') := \det[H' - E' I_{3\times3}] = &-E'^3 + (r' + s' + t')E'^2 \\ &+(-r's' - s't' - t'r' + |\xi|^2 + |\zeta|^2 + |\kappa|^2)E' \\ &+r's't' + \kappa\xi\zeta^* + \kappa^*\xi^*\zeta - r'|\kappa|^2 - s'|\zeta|^2 - t'|\xi|^2. \end{aligned} \tag{12}$$

On the other hand, since eigenvalues E_1' and E_2' are the roots of $P(E')$ and $E_2' = 0$ is doubly degenerate,

$$P(E') = -(E' - E_1')E'^2. \tag{13}$$

Comparing the two expressions (12) and (13) one finds:

$$E_1' = r' + s' + t' \tag{14}$$

$$-r's' - s't' - t'r' + |\xi|^2 + |\zeta|^2 + |\kappa|^2 = 0 \tag{15}$$

$$r's't' + \kappa\xi\zeta^* + \kappa^*\xi^*\zeta - r'|\kappa|^2 - s'|\zeta|^2 - t'|\xi|^2 = 0. \tag{16}$$

Furthermore, the fact that $E_2' = 0$ is doubly degenerate implies that the rows of the matrix $H' - E_2' I_{3\times3} = H'$ must be mutually linearly dependent. Equivalently the cofactors of the matrix elements of H' must vanish. This leads to:

$$s't' - |\kappa|^2 = 0 \tag{17}$$

$$t'\xi - \zeta\kappa^* = 0 \tag{18}$$

$$s'\zeta - \xi\kappa = 0 \tag{19}$$

† Strictly speaking the sign of E_1' is also a conventional choice.

$$r't' - |\zeta|^2 = 0 \tag{20}$$
$$r'\kappa - \xi^*\zeta = 0 \tag{21}$$
$$r's' - |\xi|^2 = 0. \tag{22}$$

In view of these equations, (15) and (16) are satisfied automatically. Moreover, either at most two of the parameters r', s' and t' vanish or all of them have the same sign as E_1'. Note that, as a result of equation (14) and the non-degeneracy requirement $E_1' \neq 0$, r', s' and t' cannot vanish simultaneously.

Equations (17)–(22) are indeed not independent. They can be reduced to the following four equations:

$$\xi = \sqrt{r's'}\mathrm{e}^{\mathrm{i}\gamma} \tag{23}$$
$$\zeta = \sqrt{r't'}\mathrm{e}^{\mathrm{i}\eta} \tag{24}$$
$$\kappa = \sqrt{s't'}\mathrm{e}^{\mathrm{i}\theta} \tag{25}$$
$$r's't'[\mathrm{e}^{\mathrm{i}\eta} - \mathrm{e}^{\mathrm{i}(\gamma+\theta)}] = 0. \tag{26}$$

Thus there are five independent real parameters. In addition, one knows that a rescaling of the Hamiltonian by a non-zero real function of the variables leaves the eigenvectors unchanged. Hence as far as the geometric phases are concerned, one may reduce the number of real parameters to four.

In the calculation of the geometric phases, I shall not explicitly perform this reduction. Hence the results will be valid for arbitrary 3×3 Hamiltonians. The final expressions for the geometric phases (connection one-forms), however, are expected to be invariant under the simultaneous scaling of the matrix elements of the Hamiltonian. In view of equations (23)–(25), this means that they must only involve the ratios of the parameters r', s' and t'.

Therefore the true parameters of this system are the ratios of r', s' and t', and the angles γ and θ. In fact it is not difficult to show that these parameters yield a local coordinate representation of $\mathbb{C}P^2$.

To see this let me recall the homogeneous coordinates on $\mathbb{C}P^2$, [13]:

$$(z^1, z^2, z^3) \equiv \left(1, \frac{z^2}{z^1}, \frac{z^3}{z^1}\right) \equiv \left(\frac{z^2}{z^1}, \frac{z^3}{z^1}\right) =: (\rho_{21}\mathrm{e}^{\mathrm{i}\phi_{21}}, \rho_{31}\mathrm{e}^{\mathrm{i}\phi_{31}}) \tag{27}$$

corresponding to the local patch $O_1 \subset \mathbb{C}P^2$ defined by $z^1 \neq 0$. In (27), $z^\mu \in \mathbb{C}$, $\rho_{\mu 1} := |z^\mu/z^1|$, and $\phi_{\mu 1} := |z^1|z^\mu/(|z^\mu|z^1)$, with $\mu = 1, 2, 3$.

The analogy with the parameters of the doubly degenerate Hamiltonian H' can then be expressed by $z^1 = \xi$, $z^2 = \zeta$, and $z^3 = \kappa$. In terms of the real variables one has

$$\rho_{21} = \sqrt{\frac{t'}{s'}} \qquad \rho_{31} = \sqrt{\frac{t'}{r'}} \qquad \phi_{21} = -(\theta + 2\gamma) \qquad \phi_{31} = \theta - \gamma \tag{28}$$

where the patch O_1 is defined by $r' \neq 0$, $s' \neq 0$. Similar relations hold for the other two patches $O_2 : z^2 = \zeta \neq 0$ ($r' \neq 0$, $s' \neq 0$) and $O_3 : z^3 = \kappa \neq 0$ ($s' \neq 0, t' \neq 0$). This leaves only the case where two of the parameters r', s' and t' vanish. In this case, the Hamiltonian H' is already diagonal and the eigenvectors are $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Therefore the connection one-forms (4) vanish identically. This case can then be excluded from the true parameter space. This concludes the explicit construction of the parameter space $M = \mathbb{C}P^2$.

Next let me express the necessary and sufficient conditions for the existence of a doubly degenerate eigenvalue in terms of the parameters of the original Hamiltonian H. Clearly the case $\xi = \zeta = \kappa = 0$ is trivial. This leaves one with five distinct cases. Namely,

(1) $\xi \neq 0$, $\zeta \neq 0$ and $\kappa \neq 0$. In this case equations (23)–(26) may be used to show:

$$\frac{\xi\kappa}{\zeta} \in \mathbb{R}^+ \tag{29}$$

$$r' = \pm\left|\frac{\xi\zeta}{\kappa}\right| \qquad s' = \pm\frac{\xi\kappa}{\zeta} \qquad t' = \pm\left|\frac{\zeta\kappa}{\xi}\right|. \tag{30}$$

Equations (11) and (30), then lead to

$$E_n = r \mp \left|\frac{\xi\zeta}{\kappa}\right| = s \mp \frac{\xi\kappa}{\zeta} = t \mp \left|\frac{\zeta\kappa}{\xi}\right|. \tag{31}$$

The last two equations together with (29) are equivalent to equations (23)–(26). They serve as the necessary and sufficient condition for the double degeneracy of E_2. More symmetric expressions for E_1 and E_2 are:

$$E_1 = \frac{1}{3}\left[r + s + t \pm 2\left(\left|\frac{\xi\zeta}{\kappa}\right| + \frac{\xi\kappa}{\zeta} + \left|\frac{\zeta\kappa}{\xi}\right|\right)\right] \tag{32}$$

$$E_2 = \frac{1}{3}\left[r + s + t \mp \left|\frac{\xi\zeta}{\kappa}\right| + \frac{\xi\kappa}{\zeta} + \left|\frac{\zeta\kappa}{\xi}\right|\right]. \tag{33}$$

Note also that in equations (30)–(33) either the top or the bottom sign must be chosen. Both choices are physically equivalent.

(2) $\xi = \zeta = 0$ and $\kappa \neq 0$. In this case $r' = 0$. Hence

$$E_2 = r \qquad s' = s - r \qquad t' = t - r \qquad E_1 = -r + s + t. \tag{34}$$

Furthermore, the necessary and sufficient condition for the double degeneracy of E_2 is

$$|\kappa|^2 = (s - r)(t - r). \tag{35}$$

(3) $\xi = \kappa = 0$ and $\zeta \neq 0$. In this case $s' = 0$ and one has:

$$E_2 = s \qquad r' = r - s \qquad t' = t - s \qquad E_1 = r - s + t. \tag{36}$$

The necessary and sufficient condition for the occurrence of a double degeneracy is

$$|\zeta|^2 = (r - s)(t - s). \tag{37}$$

(4) $\zeta = \kappa = 0$ and $\xi \neq 0$. In this case $t' = 0$ and one has:

$$E_2 = t \qquad r' = r - t \qquad s' = s - t \qquad E_1 = r + s - t. \tag{38}$$

The necessary and sufficient condition for the occurrence of a double degeneracy is

$$|\xi|^2 = (r - t)(s - t). \tag{39}$$

(5) Only one of the parameters ξ, ζ, κ is zero. In this case, eigenvalues of H cannot be doubly degenerate.

3. Connection one-forms

In order to compute the connection one-forms associated with the eigenvalues of the Hamiltonian, one must consider the first four cases of the above list separately. In terms of the local coordinate patches O_μ of the parameter space $M = \mathbb{C}P^2$, these cases correspond to

(1) the intersection $O_1 \cap O_2 \cap O_3$ in which $r' \neq 0$, $s' \neq 0$ and $t' \neq 0$
(2) the subset $O_3 - O_1 \cap O_2 \cap O_3$, in which $r' = 0$, $s' \neq 0$ and $t' \neq 0$
(3) the subset $O_2 - O_1 \cap O_2 \cap O_3$, in which $s' = 0$, $r' \neq 0$ and $t' \neq 0$

(4) the subset $O_1 - O_1 \cap O_2 \cap O_3$, in which $t' = 0$, $r' \neq 0$ and $s' \neq 0$ respectively.

The computation of the connection one-forms for all these four cases involves using equations (23)–(26) to obtain the eigenvectors of the Hamiltonian H'. This is indeed quite straightforward. Having obtained the eigenvectors, one then computes the connection one-forms using equation (4).

Case 1. $r' \neq 0$, $s' \neq 0$, $t' \neq 0$. The eigenvectors are given by:

$$\begin{aligned} |1\rangle &= N_1 \begin{pmatrix} \sqrt{r'}\,\mathrm{e}^{-\mathrm{i}\gamma} \\ \sqrt{s'} \\ \sqrt{t'}\,\mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \qquad |2,1\rangle = N_2 \begin{pmatrix} -\sqrt{s'}\,\mathrm{e}^{-\mathrm{i}\gamma} \\ \sqrt{r'} \\ 0 \end{pmatrix} \\ |2,2\rangle &= N_1 N_2 \begin{pmatrix} \sqrt{r't'}\,\mathrm{e}^{-\mathrm{i}\gamma} \\ \sqrt{s't'} \\ -(r'+s')\mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \end{aligned} \tag{40}$$

where $N_1 := (r' + s' + t')^{-1/2} = E_1^{'-1/2}$ and $N_2 := (r' + s')^{-1/2}$.

Substituting these equations in (4), and performing the necessary algebra one finds:

$$A_1 = \frac{\mathrm{d}\gamma}{1 + \frac{s'+t'}{r'}} + \frac{\mathrm{d}\theta}{1 + \frac{t'}{r'+s'}} \tag{41}$$

$$A_2 = \begin{pmatrix} \dfrac{\mathrm{d}\gamma}{1 + \frac{r'}{s'}} & \dfrac{\left[\mathrm{i}\left(\frac{r'}{2s'}\right)\mathrm{d}\left(\frac{s'}{r'}\right) - \mathrm{d}\gamma\right]\mathrm{e}^{\mathrm{i}\gamma}}{\sqrt{\frac{(r'+s')^2(r'+s'+t')}{r's't'}}} \\ \dfrac{\left[-\mathrm{i}\left(\frac{r'}{2s'}\right)\mathrm{d}\left(\frac{s'}{r'}\right) - \mathrm{d}\gamma\right]\mathrm{e}^{-\mathrm{i}\gamma}}{\sqrt{\frac{(r'+s')^2(r'+s'+t')}{r's't'}}} & -\dfrac{\mathrm{d}\gamma}{1 + \frac{r's'}{(r'+s')^2+r't'}} - \dfrac{\mathrm{d}\theta}{1 + \frac{t'}{r'+s'}} \end{pmatrix}. \tag{42}$$

Case 2. $r' = 0$, $s' \neq 0$ and $t' \neq 0$. In this case the eigenvectors are:

$$|1\rangle = N_3 \begin{pmatrix} 0 \\ s' \\ \sqrt{s't'}\,\mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \qquad |2,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |2,2\rangle = N_3 \begin{pmatrix} 0 \\ \sqrt{s't'} \\ -s'\mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \tag{43}$$

where $N_3 := [s'(s' + t')]^{-1/2}$. These equations together with equation (4) yield:

$$A_1 = \frac{-\mathrm{d}\theta}{1 + \frac{s'}{t'}} \tag{44}$$

$$A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left(\frac{-\mathrm{d}\theta}{1 + \frac{t'}{s'}} \right). \tag{45}$$

Note that in this case A_2 also leads to an Abelian geometrical phase.

Case 3: $s' = 0$, $r' \neq 0$ and $t' \neq 0$. In this case, the expressions for the eigenvectors and the connection one-forms can be obtained from the results of case 2. This is easily seen by the following relationship between the Hamiltonian $H'_{(3)}$ for this case and the Hamiltonian $H'_{(2)}$ for case 2:

$$H'_{(3)} = T_1\, H'_{(2)}\, T_1^{-1}|_{r' \to s', \theta \to -(\theta+\gamma)} \tag{46}$$

with

$$T_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = T_1^{-1}.$$

In view of equation (46), the eigenvectors and the connection one-forms are related according to

$$|n, a\rangle_{(3)} = T_1 |n, a\rangle_{(2)}|_{r' \to s', \theta \to -(\theta+\gamma)} \qquad (47)$$

$$A_{n_{(3)}} = A_{n_{(2)}}|_{r' \to s', \theta \to -(\theta+\gamma)} \qquad (48)$$

where the subscripts (2) and (3) correspond to cases 2 and 3.

Case 4. $t' = 0$, $r' \neq 0$ and $s' \neq 0$. The situation is analogous to case 3. Again one can use the relations

$$H'_{(4)} = T_2\, H'_{(2)}\, T_2^{-1}|_{t' \to r', \theta \to -\gamma} \qquad (49)$$

with

$$T_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = T_1^{-1}$$

to read off the expressions for the eigenvectors and the connection one-forms from equations (43)–(45), namely

$$|n, a\rangle_{(4)} = T_2 |n, a\rangle_{(2)}|_{t' \to r', \theta \to -\gamma} \qquad (50)$$

$$A_{n_{(4)}} = A_{n_{(2)}}|_{t' \to r', \theta \to -\gamma}. \qquad (51)$$

Clearly in all four cases the connection one-forms depend only on the ratios of the parameters r', s' and t', as expected. The formulae for the connection one-forms can be expressed in terms of the parameters of the original Hamiltonian H using equations (30), (34), (36), and (38) for cases 1–4, respectively.

4. Application to multipole interactions

The results of the preceding sections are clearly applicable for arbitrary 3×3 Hamiltonians. In particular they can be used to compute non-Abelian geometric phases of spin $j = 1$ systems with a multipole interaction Hamiltonian of the form

$$H = Q^{i_1,\ldots i_n} J_{i_1} \ldots J_{i_n} \qquad (52)$$

where $Q^{i_1,\ldots i_n}$ is symmetric in its labels and J_i are angular momentum operators.

The simplest example is the dipole Hamiltonian of equation (1). It is well-known that the eigenvalues of this Hamiltonian are non-degenerate. This can also be seen as a consequence of the results of section 2.

Using the standard expressions for the matrix representation of the angular momentum operators for $j = 1$, [15], the dipole Hamiltonian (1) can be written in the form (8) with†

$$r = -t = R^3 \qquad \xi = \kappa = \frac{R^1 + \mathrm{i}R^2}{\sqrt{2}} \qquad \zeta = s = 0. \qquad (53)$$

Therefore either $\xi = \zeta = \kappa = 0$ or $\zeta = 0$ and $\xi \neq 0 \neq \kappa$. In the former case the Hamiltonian becomes diagonal with diagonal elements being R^3, 0, and $-R^3$, i.e. the eigenvalues are

† Here and in the following calculations I have set $\hbar = 1$ for convenience.

not doubly degenerate. The latter case is a particular example of case 5 of section 2, for which a doubly degenerate eigenvalue is again impossible.

The next simplest case is a quadrupole Hamiltonian

$$H = Q^{ij} J_i J_j. \tag{54}$$

As I mentioned earlier, these Hamiltonians have been studied for the fermionic systems in [10–12]. It is shown in [12] that for the bosonic (integer spin) systems the Abelian geometrical phases vanish in this case. The same argument does not apply for the non-Abelian phases, however. This is quite easily seen by expressing the Hamiltonian (54) in the form (8). This leads to:

$$r = t = \tfrac{1}{2}(Q^{11} + Q^{22}) + Q^{33} \qquad s = Q^{11} + Q^{22}$$
$$\xi = -\kappa = \frac{Q^{13} + \mathrm{i}Q^{23}}{\sqrt{2}} \qquad \zeta = \tfrac{1}{2}(Q^{11} - Q^{22}) + \mathrm{i}Q^{12}.$$

Therefore either $\xi \neq 0$, in which case $\zeta \neq 0$, for otherwise one has the same situation as in case 5 of section 2, or $\xi = 0$. In the latter case one can also assume that $\zeta \neq 0$, since $\xi = \zeta = 0$ corresponds to the case where the eigenvectors of the Hamiltonian are independent of the parameters and the geometric phases are trivial.

For $\zeta \neq 0 \neq \xi$ and $\zeta \neq 0 = \xi$ one has particular examples of cases 1 and 3 of section 2, respectively. Therefore, in general non-trivial geometric phases may exist.

An example of a quantum system with a quadrupole Hamiltonian (54) is the asymmetric rotor, for which $Q^{12} = Q^{13} = Q^{23} = 0$. In this case, one has $\xi = \kappa = 0$. Hence, the geometric phases are trivial.

Another potentially interesting case is when both dipole and quadrupole interactions are present, i.e.

$$H = R^i J_i + Q^{jk} J_j J_k. \tag{55}$$

For a spin $j = 1$ system one can again express this Hamiltonian in the form (8). This corresponds to:

$$\begin{aligned} r &= R^3 + \tfrac{1}{2}(Q^{11} + Q^{22}) + Q^{33} \qquad s = Q^{11} + Q^{22} \\ t &= -R^3 + \tfrac{1}{2}(Q^{11} + Q^{22}) + Q^{33} \qquad \xi = \frac{R^1 + \mathrm{i}R^2}{\sqrt{2}} + \frac{Q^{13} + \mathrm{i}Q^{23}}{\sqrt{2}} \\ \zeta &= \tfrac{1}{2}(Q^{11} - Q^{22}) + \mathrm{i}Q^{12} \qquad \kappa = \frac{R^1 + \mathrm{i}R^2}{\sqrt{2}} - \frac{Q^{13} + \mathrm{i}Q^{23}}{\sqrt{2}}. \end{aligned} \tag{56}$$

Hence, in this case all possible cases of section 2 may occur and non-trivial geometric phases may be present. In particular, for an interaction Hamiltonian of the form (55) with the quadrupole part given by the asymmetric rotor Hamiltonian, non-trivial geometric phases can exist.

5. Conclusion

In this paper, the connection one-form for the adiabatic non-Abelian geometrical phase of arbitrary quantum systems with a three-dimensional Hilbert space have been calculated. This has been possible due to a very simple observation that the eigenvalue problem for a 3×3 matrix can be much more easily handled if one knows that one of the eigenvalues is doubly degenerate.

The parameter space for all such systems can be easily shown to be the projective space $\mathbb{C}P^2$. This is done using the well known symmetry arguments. An explicit construction of this space in terms of the matrix elements of the Hamiltonian, however, involves solving the eigenvalue problem. Therefore it is not an easy task. The indirect but efficient method of solving the eigenvalue problem which has been used in this article, also leads to an explicit construction of the parameter space.

The results of this article may be applied for arbitrary spin $j = 1$ systems. In particular, the dipole, quadrupole, and a combination of a dipole and quadrupole Hamiltonians have been discussed. A simple example of a quadrupole Hamiltonian is that of the asymmetric rotor. The geometric phases for this system turns out to be trivial. The addition of an appropriate dipole term to the Hamiltonian of the asymmetric rotor, however, can lead to non-trivial geometric phases.

Acknowledgments

I would like to thank Bahman Darian for his patient listening to my arguments, and wish to acknowledge the support of the Killam Foundation of Canada.

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